An Analytical Formulation of Phase Noise of Signals With Gaussian-Distributed Jitter

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***Abstract*—The output of many oscillatory systems can be approximated by a stochastic square-wave signal with noise-free amplitude and Gaussian-distributed jitter. We present an analytical treatment of the phase noise of this signal with white and Lorentzian jitter spectra. With a white jitter spectrum, the phase noise is nearly Lorentzian around each harmonic. With a Lorentzian jitter spectrum, it is a sum of several Lorentzian spectra, a summation that has a $1/f^4$ shape at far-out frequencies. With a combination of the two, it has $1/f^4$ and $1/f^2$ shapes at close-in and far-out frequencies, respectively. In all cases, the phase noise at the center frequency and the total signal power are both finite. These findings will improve our understanding of phase noise and will facilitate the calculation of phase noise using time- domain jitter analysis.**

***Index Terms*—Analytical formulation, frequency stability, oscillator noise, phase jitter, phase noise.**

I. Introduction

DUE to its practical importance in communications, the frequency stability of electrical oscillators has been the object of extensive research. Several methods have been proposed for estimating the phase noise of these oscillators, often using approximations and numerical approaches [1]–[3]. Although these approaches provide significant insight, they sometimes lead to erroneous conclusions about the behavior of phase noise (especially at close-in frequencies [4]) because their approximations are not valid for the entire spectrum. Furthermore, they are often based on frequency-domain analysis, which unnecessarily complicates the formulation of phase noise, a formulation that (using jitter-phase noise relationships) can be performed through time-domain jitter analysis with fewer approximations and sometimes even analytically [5].

We present the analytical formulation of the phase noise of a noisy square wave signal with noise-free amplitude and Gaussian-distributed jitter (Fig. 1). This signal can represent both the output of a relaxation or ring oscillator and the output of a limiting amplifier fed by an arbitrary periodic signal. Such amplifiers are routinely used for suppressing amplitude noise in periodic signals. Thus, our formulation is applicable to many oscillatory systems. We consider both white and Lorentzian jitter spectra and discuss the implications of the final equations in each case. Our formulation will improve our understanding of the characteristics of phase noise at close-in frequencies and will facilitate its calculation through time-domain jitter analysis.

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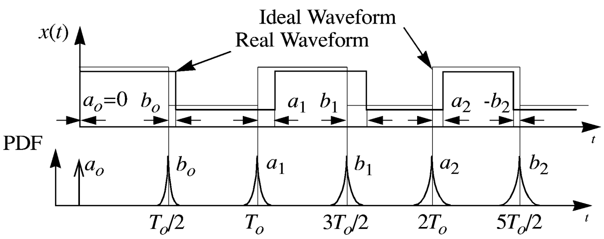

Fig. 1. Noisy signal compared to the ideal noise-free one and the probability density functions of the transitions.

II. Analytical Formulation of Phase Noise

To find an analytical expression for the phase noise of the stochastic signal $x(t)$ shown in Fig. 1, we first treat the function $x(t)$ as the sum of rectangular pulses whose start and stop times are a set of random variables

$$x(t) = \sum_{k=-\infty}^{\infty} u[t - kT_o - a_k] - u\left[t - \left(k + \frac{1}{2}\right)T_o - b_k\right] \tag{1}$$

where u represents a step function, T_o is the nominal period of the signal, and a_k and b_k are random variables characterizing the time fluctuations of the actual rise and fall instants of the signal in the kth period with respect to the anticipated rise and fall instants (Fig. 1). The statistical properties of these random variables will be discussed shortly.

We define phase noise as the single-sided power-spectral density (PSD) of the signal, normalized to the total signal power[1]. The PSD of a stochastic signal $x(t)$ is given by

$$p(j\omega) = \lim_{T\to\infty} \frac{\overline{|X_T(j\omega)|^2}}{T} \tag{2}$$

where $X_T(j\omega)$ is the Fourier transform of the windowed function $x_T(t)$, which is equal to $x(t)$ for $|t| < T$ and zero outside this interval. For simplicity, we set $T = nT_{\text{o}}$ in (2) and evaluate the limit as n goes to infinity.

Since $x(t)$ is real, $|X_T(j\omega)|^2$ can be written as

$$|X_T(j\omega)|^2 = \left[\int_{-nT_o}^{nT_o} x(t)\sin(\omega t)\,dt\right]^2 + \left[\int_{-nT_o}^{nT_o} x(t)\cos(\omega t)dt\right]^2 \tag{3}$$

[1]For a discussion about the formal definition of phase noise, please see [4].

After inserting $x(t)$ from (1), exchanging the order of integration and summation, and performing the integration, we have

$$|X_T(j\omega)|^2 = \frac{4}{\omega^2}\left[\sum_{k=-n}^{n} \sin\omega\left[\left(k+\frac{1}{4}\right)T_o + \frac{b_k+a_k}{2}\right] \times \sin\omega\left[\frac{T_o}{4}+\frac{b_k-a_k}{2}\right]\right]^2 + \frac{4}{\omega^2}\left[\sum_{k=-n}^{n} \cos\omega\left[\left(k+\frac{1}{4}\right)T_o + \frac{b_k+a_k}{2}\right] \times \sin\omega\left[\frac{T_o}{4}+\frac{b_k-a_k}{2}\right]\right]^2 \quad (4)$$

where we have also used some triangular identities.

To simplify this equation further, we note that $(\sum s_k)^2 = \sum\sum s_i s_j$. Using this identity, we convert the squared summations of (4) to double summations and combine them into one

$$|X_T(j\omega)|^2 = \frac{4}{\omega^2}\sum_{i,j=-n}^{n} \sin\omega\left[\frac{T_o}{4}+\frac{b_i-a_i}{2}\right] \times \sin\omega\left[\frac{T_o}{4}+\frac{b_j-a_j}{2}\right] \times\left\{\sin\omega\left[\left(i+\frac{1}{4}\right)T_o+\frac{b_i+a_i}{2}\right] \times \sin\omega\left[\left(j+\frac{1}{4}\right)T_o+\frac{b_j+a_j}{2}\right] + \cos\omega\left[\left(i+\frac{1}{4}\right)T_o+\frac{b_i+a_i}{2}\right] \times \cos\omega\left[\left(j+\frac{1}{4}\right)T_o+\frac{b_j+a_j}{2}\right]\right\}. \quad (5)$$

The expression inside the curvy brackets constitute the cosine of the difference between $\omega[(i+1/4)T_o+(b_i+a_i)/2]$ and $\omega[(j+1/4)T_o+(b_j+a_j)/2]$. Thus, we can rewrite (5) as

$$\begin{aligned}|X_T(j\omega)|^2 &= \frac{4}{\omega^2}\sum_{i,j=-n}^{n} \sin\omega\left[\frac{T_o}{4}+\frac{b_i-a_i}{2}\right] \times \sin\omega\left[\frac{T_o}{4}+\frac{b_j-a_j}{2}\right] \times \cos\omega\left[(i-j)T_o+\frac{b_i-b_j+a_i-a_j}{2}\right] \\ &= \frac{1}{\omega^2}\sum_{i,j=-n}^{n} \cos\omega[(i-j)T_o+b_i-b_j] + \cos\omega[(i-j)T_o+a_i-a_j] - \cos\omega\left[\left(i-j-\frac{1}{2}\right)T_o+a_i-b_j\right] - \cos\omega\left[\left(i-j+\frac{1}{2}\right)T_o+b_i-a_j\right] \end{aligned} \quad (6)$$

where, once again, we have used some triangular identities to arrive at the final form of (6).

The terms inside this double summation are only functions of $i-j$ and not of i or j alone; there is no preferred time spot for the oscillator. We can then replace $b_i - b_j$ by $c_{(i-j)}$, where $c_{(i-j)}$ is the random variable characterizing the fluctuations in the duration of the total time of $(i-j)$ consecutive periods. By the same token, $a_i - a_j, a_i - b_j$, and $b_i - a_j$ can be replaced by $d_{(i-j)}, e_{(i-j)}$ and $f_{(i-j)}$, respectively. Hence, the expression inside the double summation of (6) is only a function of $i-j$. This summation can then be simplified as follows:

$$|X_T(j\omega)|^2 = \frac{1}{\omega^2}\sum_{k=-2n}^{2n}(2n+1-|k|) \times\left\{\cos\omega[kT_o+c_k]+\cos\omega[kT_o+d_k] - \cos\omega\left[\left(k-\frac{1}{2}\right)T_o+e_k\right] - \cos\omega\left[\left(k+\frac{1}{2}\right)T_o+f_k\right]\right\}. \quad (7)$$

To calculate the expected value of (7), we need to know the statistical properties of c_k, d_k, e_k, and f_k. Due to the linearity of the expected value operator, the expected value of (7) is the sum of the expected values of the individual terms. Since each term is only a function of one random variable, we do not need to have any information about the correlation factors between the random variables involved in this expression. For calculating the expected value of (7) it is sufficient to have the statistical properties of each of the $c_k s, d_k s, e_k s$, and $f_k s$ as stand-alone random variables. Each of these random variables is the sum of several independent, zero-mean Gaussian-distributed random variables, each of which characterizes the jitter in one half- period of oscillation. Consequently, c_k, d_k, e_k, and f_k are all zero-mean, Gaussian random variables as well. We call the variances of these random variables $\sigma_{ck}^2, \sigma_{dk}^2, \sigma_{ek}^2$, and σ_{fk}^2, respectively.

We can now find the expected value of $|X_T(jw)|^2$ if we note that for a zero-mean, Gaussian random variable p, the expected value of $\cos(ap+b)$ is given by $\exp(-a^2\sigma_p^2/2)\cos b$, where σ_p^2 is the variance of p. Thus, the expected value of (7) is

$$\overline{|X_T(j\omega)|^2} = \frac{1}{\omega^2}\sum_{k=-2n}^{2n}(2n+1-|k|)\left[e^{-\frac{\omega^2\sigma_{ck}^2}{2}}\cos\omega kT_o + e^{-\frac{\omega^2\sigma_{dk}^2}{2}}\cos\omega kT_o - e^{-\frac{\omega^2\sigma_{ek}^2}{2}}\cos\omega\left(k-\frac{1}{2}\right)T_o - e^{-\frac{\omega^2\sigma_{fk}^2}{2}}\cos\omega\left(k+\frac{1}{2}\right)T_o\right]. \quad (8)$$

To find an analytical expression for phase noise, we use (8) in (2), set $T = nT_o$ and evaluate the summation. It can be shown that the part of the summation that includes $(1+|k|)$ is finite, so it vanishes when we evaluate the limit. Consequently, PSD can be written as

$$p(j\omega) = \frac{2}{\omega^2 T_o}\sum_{k=-\infty}^{\infty}\left\{e^{-\frac{\omega^2\sigma_{ck}^2}{2}}\cos\omega kT_o + e^{-\frac{\omega^2\sigma_{dk}^2}{2}}\cos\omega kT_o - e^{-\frac{\omega^2\sigma_{ek}^2}{2}}\cos\omega\left[k-\frac{1}{2}\right]T_o - e^{-\frac{\omega^2\sigma_{fk}^2}{2}}\cos\omega\left[k+\frac{1}{2}\right]T_o\right\}. \quad (9)$$

To evaluate the summation in (9), $\sigma_{ck}^2, \sigma_{dk}^2, \sigma_{ek}^2$, and σ_{fk}^2 should be expressed as functions of k. The $c_k s$ and $d_k s$ are the random variables characterizing the fluctuations of the total time of $|k|$ consecutive periods. Their variances can be replaced by $\sigma_{ck}^2 = \sigma_{dk}^2 = \sigma_{|kT_o|}^2$ where $\sigma_{|kT_o|}^2$ is the variance of the duration of this time interval. The $e_k s$ and $f_k s$ are the random variables characterizing the fluctuations of the total time of k consecutive periods minus and plus a half-period respectively. The variance of these random variables can be denoted by $\sigma_{ek}^2 = \sigma_{|(k-1/2)T_o|}^2$ and $\sigma_{fk}^2 = \sigma_{|(k+1/2)T_o|}^2$, respectively.

We can now make an approximation which will facilitate the calculation of phase noise with a Lorentzian jitter spectrum. At small offset frequencies, the fast variations of phase are not important. Thus, we can assume that the jitter occurs entirely in the second half-period and the duration of the first half-period is a deterministic variable. With this assumption, one can verify that $\sigma_{ck}^2 = \sigma_{dk}^2 = \sigma_{|kT_o|}^2 \approx \sigma_{ek}^2 \approx \sigma_{fk}^2$. We can then simplify (9) to

$$\begin{aligned} p(j\omega) &= \frac{4\left(1-\cos\frac{\omega T_o}{2}\right)}{\omega^2 T_o} \sum_{k=-\infty}^{\infty} e^{-\frac{\omega^2 \sigma_{|kT_o|}^2}{2}} \cos \omega k T_o \\ &= \frac{8}{\omega^2 T_o} \sum_{k=-\infty}^{\infty} e^{-\frac{\omega^2 \sigma_{|kT_o|}^2}{2}} \cos \omega k T_o \end{aligned} \tag{10}$$

where the second equality is because $\cos(\omega T_o/2) \approx -1$ around the fundamental frequency. We use (10) for calculating phase noise with a Lorentzian jitter spectrum. With a white jitter spectrum, however, this approximation is not necessary; in that case, we can find phase noise in its exact form.

A. Phase Noise With a White Jitter Spectrum

A white jitter spectrum is normally observed in oscillators, which have no colored noise sources and have poles only at frequencies considerably larger than the offset frequency at which we calculate phase noise [4]. With this jitter spectrum, the fluctuations of the duration of different periods are mutually independent. Thus, $\sigma_{ck}^2 = \sigma_{dk}^2 = |k|\overline{(\Delta T_o)^2}$, where $\overline{(\Delta T_o)^2}$ is the variance of the duration of one period. By the same token, $\sigma_{ek}^2 = |k\overline{(\Delta T_o)^2} - \overline{(\Delta \tau_1)^2}|$ and $\sigma_{fk}^2 = |k\overline{(\Delta T_o)^2} + \overline{(\Delta \tau_1)^2}|$, where $\overline{(\Delta \tau_1)^2}$ is the variance of the duration of the first half of the period, which is assumed independent of the variance of the duration of the second half of the period, $\overline{(\Delta \tau_2)^2}$. Using these equations in (9), we get the closed-form expression for the PSD of the signal, shown in (11) at the bottom of the page. The phase noise is defined as the PSD normalized to the total power of the carrier. We assume that the total power of the carrier is not greatly affected by noise. The total power of the first harmonic of this oscillator is $2/\pi^2$. The phase noise at an offset frequency of $\Delta\omega = |\omega - (2\pi)/T_o|$ is then given by (12) shown at the bottom of the page. Using (12), it can be shown that the phase noise has a nearly Lorentzian shape around each harmonic. Around the first harmonic, phase noise can be approximated by

$$\mathrm{PN}(\Delta f) = \frac{f_o^3 \overline{(\Delta T_o)^2}}{\left(\pi f_o^3 \overline{(\Delta T_o)^2}\right)^2 + (f - f_o)^2}. \tag{13}$$

It has been shown that the phase noise of a quasisinusoidal signal can also be approximated by a Lorentzian spectrum [6]. Equation (13) extends those results to rectangular pulses. This result is also consistent with other theoretical work on oscillatory systems [7], [8]. In the following section, we numerically show that this expression provides a high-accuracy prediction of the phase noise up to large offset frequencies.

B. Phase Noise With a Lorentzian Jitter Spectrum

A Lorentzian jitter spectrum is normally observed in oscillators, which have a dominant Lorentzian noise source and have poles only at frequencies considerably larger than the offset frequency at which we calculate phase noise [4]. With this jitter spectrum, the jitter autocorrelation function is

$$\overline{(a_i - a_{i-1}) \cdot (a_j - a_{j-1})} = A_\theta e^{\frac{-|i-j|T_0}{\theta}}. \tag{14}$$

In this case, we calculate the phase noise using (10). This approach requires us to first calculate the variance of the duration of k consecutive periods, which can be written as

$$\begin{aligned} \sigma_{Tk}^2 &= \overline{\left(\sum_{i=1}^{k}(a_i - a_{i-1})\right)^2} \\ &= \sum_{i=1}^{k}\sum_{j=1}^{k} \overline{(a_i - a_{i-1}) \cdot (a_j - a_{j-1})}. \end{aligned} \tag{15}$$

Using (14) in (15), we get

$$\sigma_{Tk}^2 = kA_\theta + 2A_\theta \sum_{i=1}^{k}(k-i)e^{-\frac{iT_o}{\theta}} \tag{16}$$

which after evaluating the summation, gives

$$\sigma_{Tk}^2 = C_\theta + D_\theta k + E_\theta e^{-\frac{kT_o}{\theta}} \tag{17}$$

$$p(j\omega) = \left[8\sinh\frac{\omega^2\overline{(\Delta T_o)^2}}{4}\right] \frac{\left\{\cosh\frac{\omega^2\overline{(\Delta T_o)^2}}{4} - \cos\frac{\omega T_o}{2}\cdot\cosh\left[\frac{\omega^2\overline{(\Delta T_o)^2}}{4} - \frac{\omega^2\overline{(\Delta \tau_1)^2}}{2}\right]\right\}}{\omega^2 T_o\left[\cosh\frac{\omega^2\overline{(\Delta T_o)^2}}{2} - \cos\omega T_o\right]} \tag{11}$$

$$\mathrm{PN}(j\omega) = \sinh\frac{\omega^2\overline{(\Delta T_o)^2}}{4} \frac{\left\{\cosh\frac{\omega^2\overline{(\Delta T_o)^2}}{4} - \cos\frac{\omega T_o}{2}\cdot\cosh\left[\frac{\omega^2\overline{(\Delta T_o)^2}}{4} - \frac{\omega^2\overline{(\Delta \tau_1)^2}}{2}\right]\right\}}{f^2 T_o\left[\cosh\frac{\omega^2\overline{(\Delta T_o)^2}}{2} - \cos\omega T_o\right]} \tag{12}$$

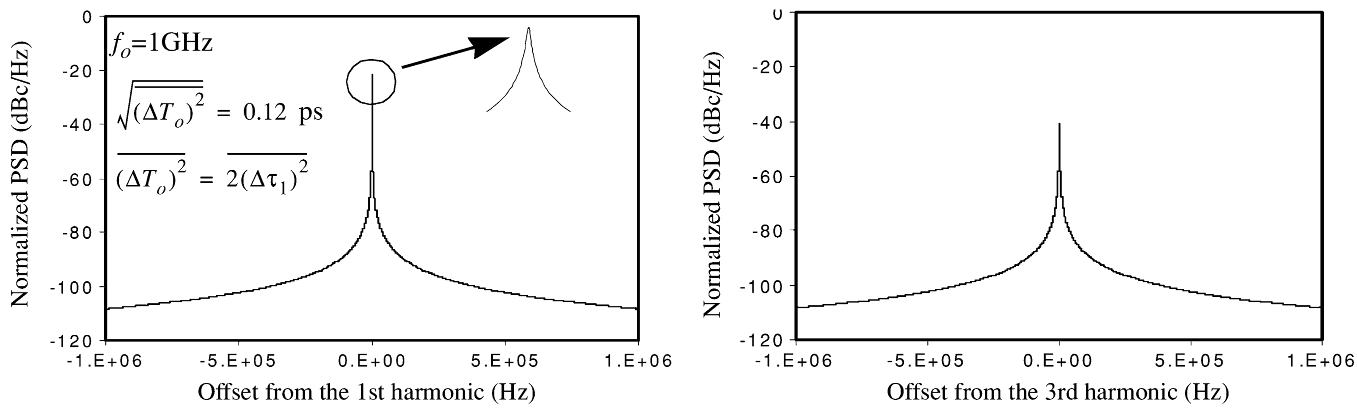

Fig. 2. PSD of the signal normalized to the total power of the first harmonic around the first and the third harmonics. These graphs are for a signal with a white jitter spectrum. By definition, 2(a) demonstrates the phase noise around the first harmonic.

where

$$C_\theta = -2A_\theta \frac{e^{-\frac{T_o}{\theta}}}{\left(1 - e^{-\frac{T_o}{\theta}}\right)^2} \tag{18}$$

$$D_\theta = A_\theta \left(1 + \frac{2e^{-\frac{T_o}{\theta}}}{\left(1 - e^{-\frac{T_o}{\theta}}\right)}\right) \tag{19}$$

$$E_\theta = 2A_\theta \frac{e^{-\frac{T_o}{\theta}}}{\left(1 - e^{-\frac{T_o}{\theta}}\right)^2}. \tag{20}$$

Using (17) in (10), we can calculate the PSD of the signal as

$$p(j\omega) = \frac{8}{\omega^2 T_o} \sum_{k=-\infty}^{\infty} e^{-\frac{\omega^2\left(C_\theta + D_\theta k + E_\theta e^{-\frac{kT_o}{0}}\right)}{2}} \cos\omega k T_o. \tag{21}$$

After expanding $\exp(-\omega^2 E_\theta \exp(-kT_o/\theta)/2)$ to its power series, (21) becomes

$$p(j\omega) = \frac{8}{\omega^2 T_o} \sum_{k=-\infty}^{\infty} e^{-\frac{\omega^2(C_\theta + D_\theta k)}{2}} \times \left(\sum_{k'=0}^{\infty} \left(-\frac{\omega^2}{2} E_\theta\right)^{k'} \frac{e^{-\frac{k' k T_o}{\theta}}}{k'!}\right) \cos\omega k T_o. \tag{22}$$

To simplify it further, we change the order of summations to get

$$p(j\omega) = \frac{8}{\omega^2 T_o} e^{\frac{\omega^2 C_\theta}{2}} \sum_{k'=0}^{\infty} \frac{\left(-\frac{\omega^2 E_\theta}{2}\right)^{k'}}{k'!} \times \sum_{k=-\infty}^{\infty} e^{-\frac{\omega^2}{2}\left(D_\theta + \frac{2k' T_0}{\omega^2 \theta}\right)k} \cos\omega k T_o. \tag{23}$$

The inner summation represents a Lorentzian PSD shown in (13) with $\overline{T^2}_{oeq} = D_\theta + (2k'T_o)/(\omega^2\theta)$. We then see that the PSD of the signal with Lorentzian jitter spectrum is a summation of several Lorentzian functions

$$p(j\omega) = e^{-\frac{\omega^2 C_\theta}{2}} \sum_{k'=0}^{\infty} \frac{\left(-\frac{\omega^2 E_\theta}{2}\right)^{k'}}{k'!} \times \frac{f_0^3\left(D_\theta + \frac{2k'T_o}{\omega^2\theta}\right)}{\left(\pi f_o^3\left(D_\theta + \frac{2k'T_o}{\omega^2\theta}\right)\right)^2 + (f - f_o)^2} \tag{24}$$

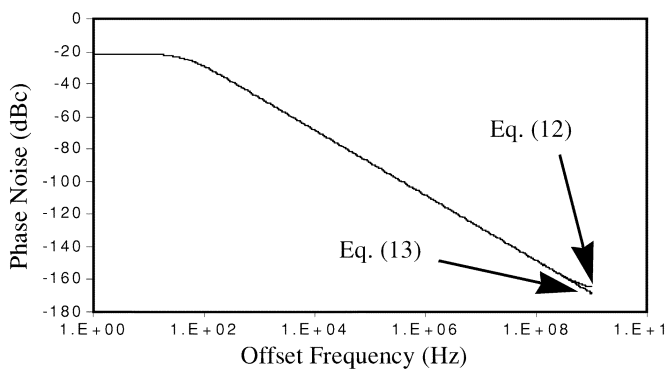

Fig. 3. Comparison of the phase noise predictions of (12) and (13) versus frequency. Design parameters are given in Fig. 2.

which after normalization gives the phase noise as

$$\mathrm{PN}(j\omega) = \frac{\sum_{k'=0}^{\infty} \frac{\left(-\frac{\omega^2 E_\theta}{2}\right)}{k'!} \frac{f_0^3\left(D_\theta + \frac{2k'T_o}{\omega^2\theta}\right)}{\left(\pi f_0^3\left(D_\theta + \frac{2k'T_o}{\omega^2\theta}\right)\right)^2 + (f - f_0)^2}}{\sum_{k'=0}^{\infty} \frac{\left(\frac{-\omega^2 E_\theta}{2}\right)^{k'}}{k'!}}. \tag{25}$$

We use (12) and (25) to discuss the general characteristics of phase noise in the next section.

III. Numerical Results and Discussion

Fig. 2 shows the normalized PSD of $x(t)$ with a white jitter spectrum around the first and third harmonics as predicted by (12). The PSD has local peaks at odd harmonics of the fundamental frequency as expected. The phase noise around the first harmonic is shown in a frequency span of 2 MHz. The phase noise is −108 dBc/Hz at 1 MHz offset frequency for an oscillation frequency of 1 GHz and a rms period jitter of 0.12 ps.

Fig. 3 shows a logarithmic plot of the phase noise around the first harmonic for the same signal. The 3 dB-corner frequency is found to be 45 Hz. To investigate the accuracy of (13), we compare its prediction to that of (12) in this figure. The prediction of (12) is quite close to that of (13) for the major part of the spectrum, consistent with previous work [7]. The error between the predictions of these two equations is less than 0.04 dB for offset frequencies smaller than 100 MHz.

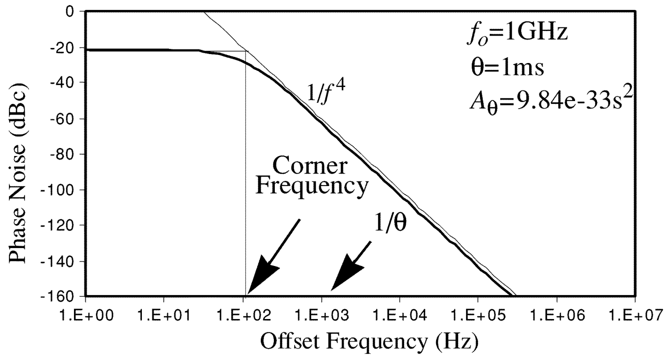

Fig. 4. Phase noise with a Lorentzian jitter spectrum.

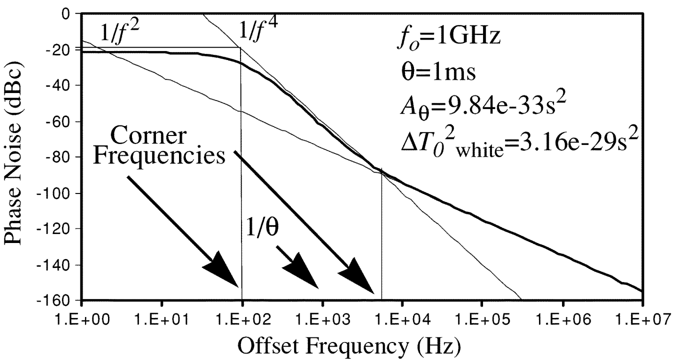

Fig. 5. Phase noise with a combination of white and Lorentzian jitter spectra, note the relative values of the corner frequencies.

Fig. 4 shows the phase noise of the signal of Fig. 1 with a Lorentzian jitter spectrum. The phase noise has a $1/f^4$ shape at far-out frequencies and is flat at close-in frequencies. The 3-dB corner frequency of the phase noise spectrum is nearly an order of magnitude smaller than that of the jitter spectrum.

With a combination of several independent jitter spectra, the superposition properties of jitter can be used to calculate the phase noise using (10), [4]. Fig. 5 shows the phase noise with a combination of white and Lorentzian jitter spectra. The phase noise has $1/f^4$and $1/f^2$shapes at close-in and far-out frequencies, respectively. We can intuitively explain this phenomenon; The Lorentzian-spectrum jitter gets smaller than the white- spectrum jitter at far-out frequencies. Thus, we expect to predominantly see the effect of white-spectrum jitter.

With very low-frequency jitter spectra, the spectrum of phase noise is dictated by jitter distribution (which is assumed Gaussian here) and not by jitter spectrum. In this case, the phase noise spectrum has a Gaussian shape. If this low frequency jitter is combined with white-spectrum jitter, the phase noise spectrum will be a Voigt line profile which graphically resembles the spectrum of Fig. 5, [9].

IV. Conclusion

We have demonstrated an analytical formulation of the phase noise of a square-wave, nearly-periodic, stochastic signal with noise-free amplitude and Gaussian-distributed jitter, a signal which can represent the output of many oscillators. This analytical formulation revealed the characteristics of phase noise with white and Lorentzian jitter spectra and a combination of the two. With a white jitter spectrum, the phase noise has a nearly Lorentzian shape. With a Lorentzian jitter spectrum, the phase noise has a $1/f^4$ shape at far-out frequencies and its corner frequency is nearly an order of magnitude smaller than the corner frequency of the jitter spectrum. With a combination of the two, the phase noise has $1/f^2$ and $1/f^4$ at far-out and close-in frequencies, respectively; it is flat around the carrier. Our formulation numerically showed the relative positions of the corner frequencies in this case.

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